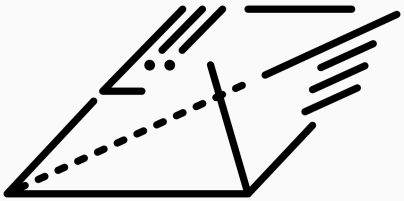
Towards an Agda formalisation of directed univalence in triangulated type theory

Samuel Toth

HOTT/UF 2026 Århus

Agda-synthetic-categories

- New Agda library
- Book-HoTT (non-cubical) foundation
- Postulated funext, univalence, pushouts and localisation HIT
- Extention of Agda's modal system
- Online resource generated with the Forester system
- Currently under active development by myself and Fredrik Bakke



agda-synthetic-categories.toth.co.uk

Table of Contents

Simplicial type theory without a strict interval

The category of spaces

Simplicial type theory without a strict interval

Recap: Simplicial type theory [RS17]

- Add an interval *pretype* $(\mathbb{I}, 0, 1, \wedge, \vee)$ - structure of a totally-ordered bounded distributive lattice
- Define synthetic momorphisms in each type:
  $\mathsf{Hom}_A(a, b) := \Sigma_{f:A^{\mathbb{I}}}(f(0) \equiv a) \times (f(1) \equiv b)$ (extension types)
- Functions are automatically functorial, function composition induces
  $f_* : \mathsf{Hom}(a, b) \to \mathsf{Hom}(f(a), f(b))$
- Can define predicate which carves out those types which *behave* like categories and then do some category theory
- Formalisation project in the proof assistant Rzk [Nik25]

New approach to simplicial type theory [GWB24]

- Now the interval is just a normal type
  - – Lose definition equations on interval
  - – Lose extension types
  - \+ Can say more about the interval. E.g. $\forall n.(x : \Delta^n)$... or (assuming some axioms), $\mathbb{I}$ is a category
  - \+ Simplifies meta theory
  - \+ No longer require a custom proof assistant
- Introduction of modalities
  - \+ Allows us to speak of *"non-functorial"* dependency
  - \+ Crucial ingredient in the construction of the category of spaces (and of categories)
  - \- Recomplicates the meta-theory and proof assistant situation

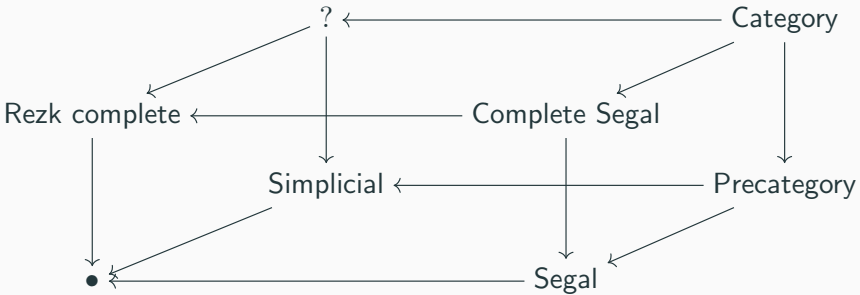
Agda synthetic categories

- Follow this second approach
- We work with minimal assumptions
- Most modules in Synthetic.Categories namespace parameterised by a distributive lattice
- In particular we don't assume the interval is totally ordered
- Instead look at those types which *see* the interval as totally ordered: A is simplicial when for all $i, j : \mathbb{I}$, $A \xrightarrow{\sim} A^{i \leq j * j \leq i}$

Modalities

- Simplicial types are an example of a higher modality ([RSS20])
- Modalities (reflective subuniverses, factorisation systems) are a key organisational tool (e.g. truncation levels)
- Prove closure properties for reflective subuniverses once and for all, and reap the rewards!
- [RSS20] also gives us a method for constructing localisations via a HIT

Menagerie of categorical reflective subuniverses



Fibrations in STT

- Covariant fibrations are an important part of original RS paper
- Morphisms in U do not necessarily correspond to families
- Consider $F : \mathbb{I} \to U := i \mapsto i = 0$, $F : \mathsf{Hom}_U(0 = 0, 1 = 0)$
- Covariant families are those families fibred in groupoids, which have a functorial action

Definition
$F : A \to$ Type is covariant if for every $\sigma : A^{\mathbb{I}}$ and $x_0 : F_{\sigma(0)}$, the type $\langle \Pi_i F_{\sigma(i)} | x_0 \rangle$ is contractible

Without extension types this turns into a section $f : \Pi_i F_{\sigma(i)}$, together with a propositional equality $f(0) = x_0$. To show this type is contractible becomes harder.

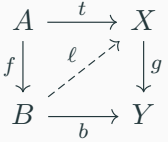
Lifting of squares

Definition
Given maps $f : A \to B$ and $g : X \to Y$, a morphism of arrows $\mathsf{Hom}_{\mathsf{Type}^{\to}}(f, g)$ is a commuting square (t, b, H):

$$\begin{array}{ccc} A & \xrightarrow{t} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[b]{} & Y \end{array}$$

Lifting of squares

Definition
Given maps $f : A \to B$ and $g : X \to Y$, a morphism of arrows $\mathsf{Hom}_{\mathsf{Type}^{\to}}(f, g)$ is a commuting square (t, b, H):



A lift of such a square is the data of:

- A diagonal map ℓ
- A filler K for the upper left triangle, and L for the lower right triangle
- Coherence data saying that K and L pasted together are equal to H

Lifting of squares

Definition
Given maps $f : A \to B$ and $g : X \to Y$, a morphism of arrows $\mathsf{Hom}_{\mathsf{Type}^{\to}}(f, g)$ is a commuting square (t, b, H):

$$\begin{array}{ccc} A & \xrightarrow{t} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[b]{} & Y \end{array}$$

- We say $f \perp g$ if for every square the space of lifts is contractible.
- If $F : A \to \mathsf{Type}$ is a family, then F is covariant iff $(\{0\} \to \Delta^1) \perp (\tilde{F} \to A)$
- This involves even more coherence data!

But, orthogonal maps enjoy many closure properties, let all maps labelled with an f be orthogonal to h:

1. $gf \perp h$ iff $g \perp h$
2. $\Sigma_i f_i \perp h$
3. $f \times f' \perp h$
4. $f_1 \amalg_{f_2} f_3 \perp h$
5. If there is a cocartesian square $f \to g$, then $g \perp h$
6. If g is a retract of f, then $g \perp h$
7. For any g, the pushout product $f \square g \perp h$

This is very useful!

Example

Lemma
Any map orthogonal to $\{0\} \to \Delta^1$ *or* $\{1\} \to \Delta^1$ *is orthogonal to* $\Lambda^2_1 \to \Delta^2$.

Proof.
It can be computed that the pushout product of $(\{1\} \to \Delta^1) \square (\{0\} \to \Delta^1)$ is the inclusion $\Lambda^2_1 \to \mathbb{I}^2$. Then we see that $\Lambda^2_1 \to \Delta^2$ is a retract of this map. $\square$

The category of spaces

What do we want?

For small maps $E \to B$, univalence tells us that we have a unique pullback square:

$$\begin{array}{ccc} E & \longrightarrow & U^\bullet \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & U \end{array}$$

Definition
If $\mathcal{F}$ is some class of maps, then a map $U^\bullet \to U$ is said to classify $\mathcal{F}$ if it is terminal in the subcategory of $\mathrm{Type}^{\to}$ with objects maps in $\mathcal{F}$, and morphisms the cartesian squares

Example

- U classifies small maps
- Prop_U classifies small embeddings

The category of spaces can be characterised as being the classifier for left fibrations.

LOPS construction

- The technique used in [GWB24] to construct the classifier for left fibration is inspired by "the LOPS construction" [LOPS18] for constructing internal universes, which makes use of the fact that the interval is a tiny type
- For tiny types to behave nicely it is important that we are in a presheaf category over a base with finite products – hence the importance of foregoing the simplicial axiom
- In [LOPS18] they also prove a no-go theorem which demonstrates the need for the $\flat$ modality, without which it would not even be possible to postulate a type with the correct universal property

Tiny types

Definition ([GWB24, Axiom 3])
We say that a type T is tiny if for each crisp type A, there is a type A_T and a map $\varepsilon_A : (A_T)^T \to A$, such that for any crisp type Γ, the induced map $\flat(\Gamma \to A_T) \to \flat(\Gamma^T \to A)$ is an equivalence. The operator $-_T$ is called the 'amazing right adjoint'.

Example
$\mathbf{1}$ is always a tiny type

Constructing classifying maps

Notice that in both of our examples of classifiers, the class of maps being classified was given by a fibrewise property: $f \in \mathcal{F}$ iff $\forall x.\, \mathsf{fib}_f(x) \in \mathcal{F}$. We generalise this notion:

Definition
A class of maps $\mathcal{F}$ is X-relative for some type X if $f : A \to B$ is in $\mathcal{F}$ iff each pullback $\sigma^* f \in \mathcal{F}$ for any $\sigma : X \to B$

Example

- The classes of left or right fibrations are $\mathbb{I}$-relative
- More generally, if $\mathcal{R}$ is the class of maps right orthogonal to some map $A \to B$, then $\mathcal{R}$ is B relative

Constructing classifying maps

Theorem
Given a tiny type T, and a class of maps $\mathcal{F}$ which is T-relative, we can construct a type U_F of amazingly fibrant types, and a map $U^{\bullet}_{\mathcal{F}} \to U_{\mathcal{F}}$ in $\mathcal{F}$, which classifies crisp U-small maps in $\mathcal{F}$.

Hence under the minimal assumptions of an interval which is tiny, we obtain types $\mathcal{S}$ and $\mathcal{S}^{op}$ which classify left and right fibrations respectively. (Without having assumed opposite categories exist in the first place)

Proof sketch.
The basic idea is to introduce dependent version of the amazing right adjoint:

$$\sqrt[T]{-} : \flat(\Gamma^T \to \mathsf{Type}) \to \flat(\Gamma \to \mathsf{Type})$$

This should satisfy the property that sections $\Pi_{a:\Gamma} \sqrt[T]{(P(a))}$ are equivalent to sections $\Pi_{a:\Gamma^T} P(a)$.

Now we may take $U_{\mathcal{F}}$ to be $\Sigma_{A:U} \sqrt[T]{A \in \mathcal{F}}$. Notice that under the root, A has type $T \to U$ (which we implicitly unstraighten to a map $\tilde{A} \to A$) and so is well-typed in $A \in \mathcal{F}$. The result then follows from the universal property of the dependent amazing right adjoint.

□

Directed univalence

We have $\mathcal{S}$, but still need to show that it is a directed univalent category:

- That the map induced by amazing covariance $\mathrm{Hom}_{\mathcal{S}}(A, B) \to (A \to B)$ is an equivalence
- That $\mathcal{S}$ forms a category: it is Segal, Rezk complete and simplicial

Current status of formalisation

The formalisation gives an almost complete proof of directed univalence that closely follows the one given by [GWB24]. The fact that $\mathcal{S}$ is a category remains to be formalised.

In addition to assuming an interval and the fact that $\mathbb{I}$ is tiny, we require the following axioms:

Axiom (Cubes separate)
$f : A \to B$ is an equivalence iff the induced map $\flat(\mathbb{I}^n \to A) \to \flat(\mathbb{I}^n \to B)$ is an equivalence for each $n : \mathbb{N}$.

Axiom (Global points of $\mathbb{I}$)
The inclusion Bool $\to \mathbb{I}$ induces an equivalence Bool $\simeq \flat\mathbb{I}$

Axiom (Quasicoherence)
We use the fact that $- = 0 : \mathbb{I} \to$ Type is contravariant – which follows from the more general quasicoherence axiom.

Current status of formalisation

The formalisation gives an almost complete proof of directed univalence that closely follows the one given by [GWB24]. The fact that $\mathcal{S}$ is a category remains to be formalised.

In addition to assuming an interval and the fact that $\mathbb{I}$ is tiny, we require the following axioms:

Axiom (Cubes separate)
$f : A \to B$ is an equivalence iff the induced map $\flat(\mathbb{I}^n \to A) \to \flat(\mathbb{I}^n \to B)$ is an equivalence for each $n : \mathbb{N}$.

Axiom (Global points of $\mathbb{I}$)
The inclusion Bool $\to \mathbb{I}$ induces an equivalence Bool $\simeq \flat\mathbb{I}$

Axiom (Quasicoherence)
We use the fact that $- = 0 : \mathbb{I} \to$ Type is contravariant – which follows from the more general quasicoherence axiom.

Remaining holes in the proof

- The glue type is defined as $Gl(A, B, f) := i \mapsto \Sigma_{b:B}(i = 0) \to \mathsf{fib}_f(b)$
  - Closure of amazing covariance under sums and pullbacks has been formalised
  - The challenge is to show $(i = 0) \to X$ is amazingly covariant when X is
  - Currently this is an additional axiom in our formalisation, but progress has been made, this hole is filled up to the lemma that cofinal maps are left anodyne
- We also currently take the fact dependent amazing right adjoints preserve propositions as an additional assumption.
  - This should follow easily from the fact they preserve pullbacks, or from a characterisation of the identity type of the non-dependent right adjoint

Future work

- Finish the formalisation of $\mathcal{S}$
- Work towards the generalisation to the category of categories [GWB26]
- Consider category of categories and profunctors (classifying exponentiable fibrations)
- ...

Thank you for listening

References

[GWB24] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz.
Directed univalence in simplicial homotopy type theory. July 2024.
DOI: https://doi.org/10.48550/arXiv.2407.09146.

[GWB26] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz.
The ∞-Category of ∞-Categories in Simplicial Type Theory.
Feb. 2, 2026. DOI: 10.48550/arXiv.2602.02218.

References ii

[LOPS18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. **"Internal Universes in Models of Homotopy Type Theory".** en. In: *LIPIcs, Volume 108, FSCD 2018* 108 (2018), 22:1–22:17. ISSN: 1868-8969. DOI: 10.4230/LIPIcs.FSCD.2018.22.

[Nik25] Nikolai Kudasov. **Rzk proof assistant.** Version v0.7.7. Oct. 3, 2025. URL: https://rzk-lang.github.io/rzk/en/v0.7.7/.

[RS17] Emily Riehl and Michael Shulman. **A type theory for synthetic ∞-categories.** May 2017. DOI: 10.48550/arXiv.1705.07442.

References iii

[RSS20] Egbert Rijke, Michael Shulman, and Bas Spitters. **"Modalities in homotopy type theory".** In: *Logical Methods in Computer Science* 16 (2020). DOI: 10.23638/LMCS-16(1:2)2020.

[Shu17] Michael Shulman. **Brouwer's fixed-point theorem in real-cohesive homotopy type theory.** en. Apr. 2017. DOI: 10.48550/arXiv.1509.07584.